THE KAZHDAN-LUSZTIG POLYNOMIAL OF A DELETION MATROID

An Honors Thesis Presented

By

ARTEM VYSOGORETS

Approved as to style and content by:

**** Tom C Braden 05/01/19 20:46 ****
Chair

**** Paul E Gunnells 05/01/19 23:37 ****
Committee Member

**** Sohrab M Shahshahani 05/02/19 11:54 ****
Honors Program Director

ABSTRACT

We develop new matroid constructions that relate the Kazhdan-Lusztig polynomial of a matroid to that of its deletion. For graphic matroids, a condition is imposed for this result to simplify to a finer equation. This equation is then used to prove a closed formula for the Kazhdan-Lusztig polynomials of a particular family of graphic matroids.

1 Introduction

In 2014, Ben Elias, Nicholas Proudfoot and Max Wakefield introduced a Kazhdan-Lusztig polynomial of a matroid in analogy with theory of Kazhdan-Lusztig polynomials for Coxeter groups [1]. A matroid is a combinatorial structure that generalizes the concept of linear independence of vectors in a vector space. The key features and examples of matroids will be presented in Section 2. Elias et al. showed that, for any matroid M, it is possible to associate a unique polynomial $P_M(t)$ with integer coefficients that satisfies certain conditions. In the same work, its coefficients are conjectured to be non-negative and form a log-concave sequence. The recursive definition of Kazhdan-Lusztig polynomials makes it difficult to find them explicitly in general and, therefore, they remain unknown for the majority of matroids.

Some progress has been made for several "nice" and symmetric matroid families. In particular, Kazhdan-Lusztig polynomials were computed for uniform (Gao et al.), thagomizer (Gedeon), wheel, whirl, and fan matroids (Lu et al.) [2, 3, 7]. For example, a graphic matroid based on a cycle C_{d+1}, a special case of a uniform matroid, has a known formula for the coefficients c_d^i of its Kazhdan-Lusztig polynomial $P_{C_{d+1}}(t)$:

$$c_d^i = \frac{1}{d-i}\binom{d+1}{i}\binom{d-i}{i+1}, \qquad 0 \leqslant i \leqslant \left\lfloor \frac{d-1}{2} \right\rfloor. \tag{1}$$

The original focus of this study was on computing Kazhdan-Lusztig polynomials associated with double-cycles $C_{m,n}$, which are graphs composed of two edge-intersecting cycles C_m and C_n, $m, n \geqslant 3$. In Section 4, we discuss the structure of the corresponding matroid $M_{m,n}$, count and classify its flats and contractions. A Mathematica program, described in Appendix A, was then designed to compute Kazhdan-Lusztig polynomials of double-cycle matroids [6]. Linear and quadratic coefficients of $P_{M_{m,n}}(t)$ for small values of m and n are displayed in Tables 1 and 2. Based on the obtained results, we proposed Theorem 4.7 that claims

$$P_{C_{m,n}}(t) = P_{C_d}(t) - tP_{C_{n-1}}(t)P_{C_{m-1}}(t), \tag{2}$$

where $d = m + n - 2$. The above formula computes Kazhdan-Lusztig polynomials of double-cycle matroids via those corresponding to cycles of sizes $m + n - 2$, $m - 1$ and $n - 1$. The first cycle is obtained by deleting the distinguished edge from $C_{m,n}$, while the other two are contractions C_m/e and C_n/e. This observation led to a more general discussion concerning Kazhdan-Lusztig polynomial of a deletion matroid M/e presented in Section 5.

In analogy with the Z-polynomial of any matroid M, we introduced a $\mathcal{Z}$-function on a lattice of flats $L(M)$ in Definition 5.4 [9]. Since flats of a deletion matroid $M\backslash e$ are obtained simply as $F\backslash e$ for every $F \in L(M)$, it was possible to understand and quantify the relation between $\mathcal{Z}$-functions of M and $M\backslash e$, which is stated in

Theorem 5.16. The main result of this study, the connection between Kazhdan-Lusztig polynomials $P_M(t)$ and $P_{M\setminus e}(t)$, follows immediately. We have

$$P_M(t) = P_{M\setminus e}(t) - tP_{M/e}(t) + \sum_{F \in \mathcal{R}_M(e)} P_{M_F}(t)[t^{\tau(F)}]t^{\tau(F)+1}P_{M^{F\setminus e}}(t), \tag{3}$$

where $\mathcal{R}_M(e)$ is a special collection of flats given in Definition 5.14 and $\tau(F)$ is a special integer value for $F \in \mathcal{R}_M(e)$. This result is applicable to any matroid M and provides a novel technique in computing Kazhdan-Lusztig polynomials of matroids by iteratively deleting its non-isthmus elements. In Section 6, we illustrate its use for edge-separable graphs (Definition 6.1), which are designed specifically for the sum in identity (3) to vanish. In Theorem 6.2, we prove that, for an edge-separable graph G with an edge e, (3) simplifies to

$$P_G(t) = P_{G\setminus e}(t) - tP_{H/e}(t)P_{K/e}(t). \tag{4}$$

In particular, this proves the conjectured formula (2) for double-cycle graphs as they satisfy the edge-separability requirement.

2 Preliminaries

This section briefly introduces matroids and related concepts that will be extensively used throughout the paper. Contents of this section, for the most part, are taken from *Matroids. A Geometric Introduction* by Gordon and McNulty [4]. We also consulted papers by Mayhew, Royle [8] and Heunen, Patta [5].

A *matroid* is a combinatorial structure that generalizes the concept of linear independence of vectors in a vector space. It is given by a finite *ground set* E and a family $\mathcal{I}$ of its subsets (called *independent sets*) that collectively satisfy certain properties referred to as *matroid axioms*. Due to their abstract definition, matroids serve as a frame for a myriad of seemingly unrelated problems that arise in different parts of discrete mathematics, optimization, and beyond. We begin by surveying two particular classes of matroids that will be most relevant in this study.

Linear and Graphic Matroids

The most widely recognized example of a matroid is a *linear* matroid, which is based on a collection of vectors from a vector space over a field and whose independent sets are simply linearly independent subcollections of these vectors. Any matroid that is isomorphic to a linear matroid for some field is called *representable*. A *graphic* matroid $M(G)$ is built on the edge set of a graph G and has its forests as independent sets. Note that graphic matroids are representable as it is possible to associate a collection of vectors to a graph such that the two induced matroids are isomorphic. We illustrate this correspondence with an example below.

Example 2.1 Let $M = M(C_4)$ be a graphic matroid corresponding to a 4-cycle with edge set $E(C_4) = \{e_{12}, e_{23}, e_{34}, e_{14}\}$. Clearly, any three or fewer edges of C_4 form a forest, so any proper subset of $E(C_4)$ is independent. Now, for each edge $e_{ij} \in E(C_4)$, consider a hyperplane in $\mathbb{R}^{|V(C_4)|} = \mathbb{R}^4$ given by $x_i - x_j = 0$ and let $\overrightarrow{n_{ij}}$ denote its normal. We claim that the resulting collection $\mathcal{C}$ of normals

$$\overrightarrow{n_{12}} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \overrightarrow{n_{23}} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \overrightarrow{n_{34}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad \overrightarrow{n_{14}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

induces a linear matroid $M' = (\mathcal{C}, \mathcal{I}')$ isomorphic to M. It is trivial to verify that any subset of three or fewer normals from $\mathcal{C}$ is linearly independent, which implies that matroids M' and M have the same independence structure and hence isomorphic under the map $e_{ij} \mapsto n_{ij}$.

In particular, we see that any graphic matroid $M = M(G)$ can be associated to a hyperplane arrangement in $\mathbb{R}^{|V(G)|}$. More on this topic can be found in lecture notes on hyperplane arrangements by Stanley [11].

Matroid Constructions

Throughout this paper, we will refer to several constructions that arise in the context of matroids. First, any matroid $M = (E, \mathcal{I})$ has a *rank* function $r\colon 2^E \to \mathbb{Z}_{\geqslant 0}$, which associates any $X \in 2^E$, with the size of the largest independent set contained in it:

$$r(X) = \max_{J \subseteq X} \left\{ |J| \mid J \in \mathcal{I} \right\}. \tag{5}$$

A closely related function is *corank* $cr\colon 2^E \to \mathbb{Z}$, which simply maps any subset $X \in 2^E$ to $r(M) - r(X)$, where $r(M)$ is a standard notation for $r(E)$. Two key properties of a rank function that are used in this study include *unit-increase* and *semimodularity*. For any $A, B \in 2^E$ and $x \in E$,

(i) Unit-increase: $r(A) \leqslant r(A \cup \{x\}) \leqslant r(A) + 1$,
(ii) Semimodularity: $r(A \cup B) + r(A \cap B) \leqslant r(A) + r(B)$.

Note that the unit-increase property ensures that corank function takes non-negative values as no subset of E has rank higher than $r(M)$.

The rank function has a clear interpretation for graphic and linear matroids. For a subset X of edges in a graphic matroid $M = M(G)$, the rank of X is the size of the spanning forest of the subgraph $H = (V(G), X)$. The rank of a collection of vectors is just the rank of a corresponding matrix.

We will now describe a few important classes of subsets of a matroid's ground set E. A *basis* is a maximal independent set, while a *circuit* is a minimal dependent (not independent) set. The terminology comes from graphic matroids, where minimal dependent sets are cycles. Any independent set is, in turn, a subset of some basis. A $flat$ is a subset $F \subseteq E$ such that for every element $x \in E\backslash F$, we have $r(F \cup \{x\}) > r(F)$. Ordered by inclusion, flats form a partially ordered set where each pair of flats has a unique supremum and a unique infimum. For this reason, the collection of all flats of a matroid M is called a *lattice of flats* and is denoted by $L(M)$. Note that $\varnothing$ and E are the unique minimal and maximal elements of $L(M)$ and, when they appear as flats, we will sometimes refer to them as $\hat{0}$ and $\hat{1}$, respectively.

Example 2.2 Consider the 4-cycle matroid $M = M(C_4)$ from Example 2.1 and denote its ground set by $E(C_4) = \{a, b, c, d\}$. Since any proper subset of $E(C_4)$ is independent, $L(M)$ must contain all possible subsets of size 2 or less. Any order 3 collection, on the other hand, is not a flat because appending the fourth edge creates a cycle. Finally, *abcd* must be in $L(M)$. Thus, the complete lattice of flats is shown as a Hasse diagram in Figure 1.

Flats of a graphic matroid $M = M(G)$ where $G = (V, E)$ correspond to edges of those subgraphs $H = (V, E')$ whose number of connected components will decrease upon addition of any edge $e \in E\backslash E'$. Similarly, for linear matroids, flats are collections of vectors F such that any other vector $v \in E\backslash F$ is not in the span of F.

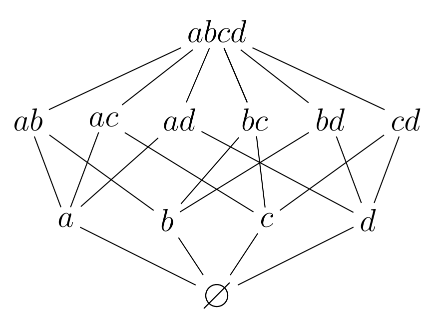

Figure 1: Hasse diagram of the lattice of flats of $M(C_4)$.

The definition of Kazhdan-Lusztig polynomial will require the *characteristic polynomial* $\chi_M(t)$ of a matroid, which is defined as

$$\chi_M(t) = \sum_{F \in L(M)} \mu(F) t^{cr(F)}, \tag{6}$$

where μ is a Möbius function of the lattice $L(M)$. Möbius values $\mu(F)$ are computed recursively as will be shown in Example 3.3.

Deletion, Restriction and Contraction

There are plenty of different ways of constructing new matroids from old. First, define an *isthmus* to be an element of the ground set that belongs to every basis and a *loop* to be an element that belongs to none. Now, given a matroid $M = (E, \mathcal{I})$ and an element $x \in E$, the *deletion* $M\backslash x$ and *contraction* M/x matroids are defined as follows:

$$\begin{aligned} M\backslash x &= (E\backslash x, \{J \mid x \notin J, J \in \mathcal{I}\}) \,, \\ M/x &= (E\backslash x, \{J\backslash x \mid x \in J, J \in \mathcal{I}\}) \,. \end{aligned} \tag{7}$$

In order to avoid empty matroids, some authors require that these operations are applied with respect to non-isthmus (for deletion) and non-loop (for contraction) elements $x \in E$. The definition of Kazhdan-Lusztig polynomials (Section 3) allows empty matroids, so we relax this condition. Repeated application of these operations with respect to elements of some subset $X \subseteq E$ yields *restriction* $M^{E\backslash X}$ and *contraction* M_X matroids, respectively. For convenience, we invert the notation and say that M^X is a restriction matroid with respect to X to mean that it is obtained by successive deletion of elements of $E\backslash X$ from M. Deletion and contraction operations are commutative, which ensures that the above matroids are well-defined. Commutativity also shows that for any $X \subseteq E$, $Y \subseteq X$ and $Z \subseteq E\backslash X$, we have

$$\left(M^X\right)^Y = M^{X \cup Y} \text{ and } \left(M_X\right)_Z = M_{X \cup Z}. \tag{8}$$

Finally, note that M_X is often referred to as *localization* matroid. Still, we will adhere to the original terminology.

The rank functions and lattices of flats of deletion, restriction and contraction matroids are closely related to those of original matroids. We will use $r^{(e)}$, r^X and r_X to denote rank functions of $M\backslash e$, M^X and M_X, respectively (corank functions will be abbreviated accordingly). Then, for any $A \subseteq E\backslash e$, $B \subseteq X$, $C \subseteq E\backslash X$ and any flat $F \in L(M)$, we have

$$L\left(M\backslash e\right) = \{F\backslash e \mid F \in L(M)\}, \tag{9}$$

$$r^{(e)}(A) = r(A); \tag{10}$$

$$L\left(M^F\right) = \{D \in L(M) \mid D \leqslant F\}, \tag{11}$$

$$r^X(B) = r(B); \tag{12}$$

$$L\left(M_F\right) = \{D\backslash F \mid D \in L(M), F \leqslant D\}, \tag{13}$$

$$r_X(C) = r(C \cup X) - r(X). \tag{14}$$

where r is a rank function of the original matroid M. These identities are direct consequences of the above definitions. We prove (13) as an example.

Proof of (13). Let D be a flat of the original matroid M such that $F \subseteq D$ and suppose that $D\backslash F$ is not a flat in M_F. Then, there must exist $x \in E\backslash F$ with $x \notin D\backslash F$ (in particular, $x \notin D$) such that $r_F\left((E\backslash F) \cup \{x\}\right) = r_F(E\backslash F)$. Then, using identity (14), we have

$$\begin{aligned} 0 &= r_F\left((D\backslash F) \cup \{x\}\right) - r_F\left(D\backslash F\right) \\ &= \big(r\left((D\backslash F) \cup \{x\} \cup F\right) - r(F)\big) - \big(r\left((D\backslash F) \cup F\right) - r(F)\big) \\ &= r(D \cup \{x\}) - r(D), \end{aligned}$$

which contradicts the fact that $D \in L(M)$. Hence, $\{D\backslash F \mid D \in L(M), F \subseteq D\}$ is a subset of $L(M_F)$. For the reverse inclusion, consider $C \in L(M_F)$ and assume $C \cup F$ is not a flat of the original matroid, so that there exists $x \notin C \cup F$ with $r\left((C \cup F) \cup \{x\}\right) \neq r(C \cup F)$. However, since D is a flat in the deletion matroid, we apply the same argument as above to derive a contradiction:

$$\begin{aligned} 0 &< r_F(C \cup \{x\}) - r_F(C) \\ &= (r((C \cup \{x\}) \cup F) - r(F)) - (r(C \cup \{x\}) - r(F)) \\ &= r\left((C \cup F) \cup \{x\}\right) - r(C \cup F). \end{aligned}$$

Therefore, $C \cup F \in L(M)$ and $F \subseteq (C \cup F)$ by choice of C. □

When restriction or contraction is performed with respect to a flat $F \in L(M)$ Hasse diagrams of $L(M^F)$ and $L(M_F)$ are represented by lower and upper intervals of F in the Hasse diagram of $L(M)$, respectively. We illustrate this geometric interpretation with the 4-cycle matroid from previous examples.

Example 2.3 Consider a flat $d \in L(M)$ of a 4-cycle matroid $M = M(C_4)$. Figure 2 displays Hasse diagrams of $L(M^d)$ and $L(M_d)$.

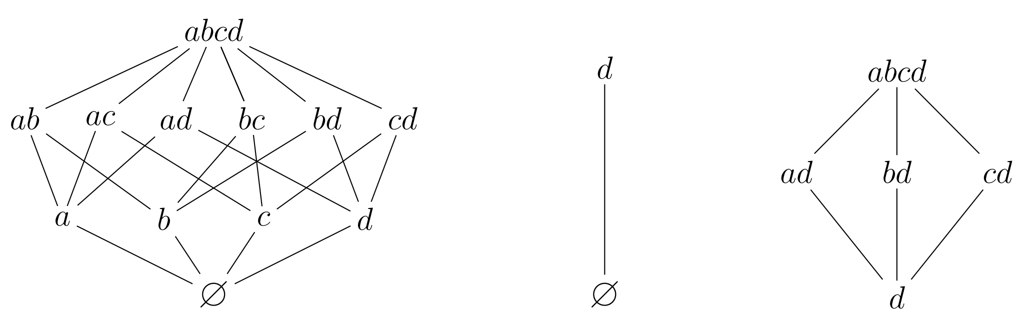

Figure 2: Hasse diagrams of $L(M)$, $L(M^d)$ and $L(M_d)$, left to right.

The contraction of a graphic matroid $M(G)$ with respect to a subset $X \subseteq E(G)$ is a matroid induced by a contraction graph G/X. Note that G/X is obtained by identifying vertices of G that are connected by edges in X. In short, $M(G)_X = M(G)/X = M(G/X)$. Similarly, the restriction $M(G)^{E\backslash X} = M(G)\backslash X$ is a matroid associated with $G\backslash X$, which is the original graph G with all edges of X removed.

Palindromic and Antipalindromic Polynomials

Before stating the definition of Kazhdan-Lusztig polynomial of a matroid, we need to introduce some more terminology. A polynomial $p(t)$ of degree n is *palindromic of degree* n if $p(t) = t^n p(t^{-1})$. Moreover, $p'(t)$ will be called *antipalindromic of degree* n when $p'(t) = -t^n p'(t^{-1})$. For example, consider

$$p(t) = t^4 + 3t^3 + 6t^2 + 3t + 1,$$
$$p'(t) = t^4 + 3t^3 + 6t^2 - 3t - 1.$$

Here, $p(t)$ is palidromic and $p'(t)$ is antipalindromic, both of degree 4.

3 Kazhdan-Lusztig Polynomial of a Matroid

In 2014, Ben Elias, Nicholas Proudfoot and Max Wakefield proved that, for every matroid M, it is possible to associate a unique (Kazhdan-Lusztig) polynomial $P_M(t)$ with integer coefficients that satisfies the following conditions:

- if $r(M) = 0$, then $P_M(t) = 1$,
- if $r(M) > 0$, then $\deg(P_M(t)) < \frac{r(M)}{2}$,
- $t^{r(M)} P_M(t^{-1}) = \sum_{F \in L(M)} \chi_{M^F}(t) P_{M_F}(t)$.

The last condition can be rewritten as

$$t^{r(M)} P_M(t^{-1}) - P_M(t) = \sum_{F \neq \varnothing} \chi_{M^F}(t) P_{M_F}(t), \tag{15}$$

because the the characterstic polynomial $\chi_{M^{\hat{0}}}(t) = 1$ by definition. Elias et al. showed that the sum on the right of (15) is antipalindromic of degree $r(M)$. Thus, the second property of Kazhdan-Lusztig polynomials ensures that $P_M(t)$ is simply a negation of the lower half of the polynomial given by that sum. Before we illustrate this argument by computing $P_M(t)$ of a 4-cycle matroid $M = M(C_4)$, we make two important observations.

Proposition 3.1 The constant term of Kazhdan-Lusztig polynomial $P_M(t)$ is always 1. This fact can be shown by induction on $r(M)$, where the base case is provided by the first condition on Kazhdan-Lusztig polynomials. Refer to Proposition 2.11 of [2] for a complete proof.

Proposition 3.2 For any matroid M with $r(M) \geqslant 2$ and a flat $F \in L(M)$ of rank $r(M)$, $r(M)-1$ or $r(M)-2$, $P_{M_F}(t) = 1$. To see this, recall that $r_F(M_F) = r(M)-r(F)$ by identity (14), so that $r_F(M_F) = 0, 1,$ or 2 for the above choice of F. Now, the degree restriction of Kazhdan-Lusztig polynomials implies that $P_{M_F}(t)$ must be constant and hence, by Proposition 3.1, we have $P_{M_F}(t) = 1$.

Note that from now on, we abuse notation and write $P_G(t)$ to refer to $P_M(t)$, where $M = M(G)$.

Example 3.3 In this example, we compute $P_{C_4}(t)$ using the above considerations. First, we need to evaluate the sum in equation (15). Recall from Example 2.2 that there is one flat of rank 0 ($\hat{0}$), four flats of rank 1 (a, b, c, d), six flats of rank 2 (all pairs ij, $i \neq j$, $i, j \in E(C_4)$) and one flat of rank 3 ($\hat{1}$). From Example 2.3 we see that lattices $L(M^F)$ of restriction matroids with respect to flats F of equal rank have the same structure and, therefore, the same characteristic polynomials $\chi_{M^F}(t)$ and Kazhdan-Lusztig polynomials $P_{M_F}(t)$. Hence, the desired sum can be written as

$$\sum_{F \neq \varnothing} \chi_{M^F}(t) P_{M_F}(t) = \chi_{M^{\hat{1}}}(t) P_{M_{\hat{1}}}(t) + 6\chi_{M^{ab}}(t) P_{M_{ab}}(t) + 4\chi_{M^a}(t) P_{M_a}(t). \qquad (16)$$

To compute the characteristic polynomials that appear above, we need to find Möbius value $\mu(F)$ for each flat. By definition, $\mu(\varnothing) = \mu(\hat{0}) = 1$ and the sum of Möbius values over all elements in the lattice is zero. Hence, we assign values recursively as shown in Figure 3. Again, due to symmetry of $L(M^F)$ for each $F \in L(M)$, flats of equal rank have the same Möbius value within the same lattice.

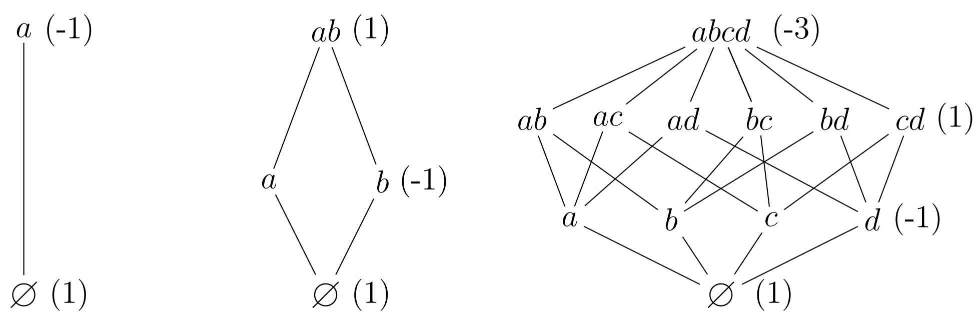

Figure 3: Lattices $L(M^a)$, $L(M^{ab})$ and $L(M)$ with Möbius values, left to right.

Since $\mu(\varnothing) = 1$, we compute $\mu(i) = -1$ for all $i \in E$, $\mu(ij) = 1$ for any distinct pair $i, j \in E$ and $\mu(abcd) = 0 - 6 + 4 - 1 = 3$. Therefore,

$$\begin{aligned} \chi_M(t) &= -3 + 6t - 4t^2 + t^3, \\ \chi_{M^{ij}}(t) &= 1 - 2t + t^2, \\ \chi_{M^i}(t) &= -1 + t. \end{aligned}$$

Now, for all nonempty flats $F \in L(M)$, $r(F) = 1, 2$ or 3, so that $P_{M_F}(t) = 1$ by Proposition 3.2. We are now ready to compute the sum in (16):

$$\begin{aligned} \sum_{F \neq \varnothing} \chi_{M^F}(t) P_{M_F}(t) &= 4(-1 + t) + 6(1 - 2t + t^2) + (-3 + 6t - 4t^2 + t^3) \\ &= t^3 + 2t^2 - 2t - 1 = t^{r(M)} P_M(t^{-1}) - P_M(t). \end{aligned} \qquad (17)$$

Since $\deg(P_M(t)) < r(M)/2 = 1.5$, it must be at most 1. It then follows that $P_M(t) = P_{C_4}(t) = 2t + 1$, which is obtained by taking negative coefficients of the lower half of (17).

This example was fairly easy to compute largely due to the size and symmetries of C_4. However, for larger matroids, the recursive nature of this approach suggests exponential increase in computational cost. For example, in order to find the Kazhdan-Lusztig polynomial of a 5-cycle using this technique, one would need to obtain $P_{C_4}(t)$ as an intermediate step.

Z-polynomial of a Matroid

Another source of recursion in the above algorithm comes from computing characteristic polynomials of restriction matroids $\chi_{M^F}(t)$. In 2017, Proudfoot, Xu and Young introduced the Z-polynomial of a matroid, which provides an alternative method of finding Kazhdan-Lusztig polynomials that does not require computing $\chi_{M^F}(t)$ explicitly [9]. The Z-polynomial is defined as

$$Z_M(t) := \sum_F t^{r(F)} P_{M_F}(t), \tag{18}$$

where $F \in L(M)$. Proudfoot et al. showed that the Z-polynomial is palindromic of degree $r(M)$. Since $\deg(P_M(t)) < r(M)/2$ by definition, $P_M(t)$ can be uniquely determined from

$$Z_M(t) - P_M(t) = \sum_{F \neq \varnothing} t^{r(F)} P_{M_F}(t) \tag{19}$$

by selecting coefficients that will yield a palindromic polynomial $Z_M(t)$ of degree $r(M)$. Given the importance of the above polynomial in practice, we will refer to it as $Z^*_M(t)$.

Example 3.4 For a graphic matroid $M = M(C_4)$ from previous examples, we immediately have

$$Z^*_M(t) = \sum_{F \neq \varnothing} t^{r(F)} P_{M_F}(t) = t^3 + 6t^2 + 4t + 1, \tag{20}$$

since $P_{M_F}(t) = 1$ for any nonempty $F \in L(M)$. Note that coefficients of $Z^*_M(t)$ represent the number of flats of a corresponding rank. Now, $P_M(t)$ is at most linear polynomial that makes $Z^*_M(t) + P_M(t)$ palindromic of degree $r(M) = 3$. Thus, $P_M(t) = 2t + 1$ as in Example 3.3.

A larger graph is required to show this method with several recursion frames. The next example uses the $\mathcal{Z}$-polynomial to compute the Kazhdan-Lusztig polynomial of a 6-cycle.

Example 3.5 Let $G = C_6$, $M = M(G)$ and denote the edge set by $E(G) = \{a, b, c, d, e, f\}$. Similarly to the case with a 4-cycle, $L(M)$ consists of all subsets of $E(G)$ except for those of size 5. All contractions with respect to flats of equal rank are isomorphic, which leads to

$$Z^*_M(t) = t^5 P_{M_{\hat{1}}}(t) + 15t^4 P_{M_{abcd}}(t) + 20t^3 P_{M_{abc}}(t) + 15t^2 P_{M_{ab}}(t) + 6tP_{M_a}(t) \tag{21}$$
$$= t^5 + 15t^4 + 20t^3 + 15t^2(2t + 1) + 6tP_{M_a}(t). \tag{22}$$

In the above equation, $P_{M_{\hat{1}}}(t) = P_{M_{abcd}}(t) = P_{M_{abc}}(t) = 1$ by Proposition 3.2. Additionally, $M_{ab} = M(G/ab))$, where G/ab is a 4-cycle and has $P_{C_4}(t) = 2t + 1$ based on Examples 3.3, 3.4. Therefore, we are left to compute the Kazhdan-Lusztig polynomial of $M' = M_a = M(C_5)$, which we do with the same strategy. Flats of M' are all subsets of $\{b, c, d, e, f\}$ except those with 4 edges. Due to the symmetries of $L(M')$, we have

$$\begin{aligned} Z^*_{M'}(t) &= t^4 P_{M'_{\hat{1}}}(t) + 10t^3 P_{M'_{bcd}}(t) + 10t^2 P_{M'_{bc}}(t) + 5tP_{M'_b}(t) \\ &= t^4 + 10t^3 + 10t^2 + 5t(2t + 1) \\ &= t^4 + 10t^3 + 20t^2 + 5t. \end{aligned} \tag{23}$$

Here, we again used the degree condition to conclude that $P_{M'_{\hat{1}}}(t) = P_{M'_{bcd}}(t) = P_{M'_{bc}}(t) = 1$. Moreover, $M'_b = M(C_5/b) = M(C_4)$, so $P_{M'_b}(t) = 2t + 1$. Now, $P_{M'}(t)$ is an at most linear polynomial that gives a palindromic polynomial of degree 4 when added to the right side of Equation (23); hence, $P_{M'}(t) = 5t + 1$. Note that $M' = M/a = M(C_6/a) = M(C_5)$, so we casually found that Kazhdan-Lusztig polynomial of a 5-cycle is $5t + 1$. We plug this result back in (22) and finally rewrite it as

$$\begin{aligned} Z^*_M(t) &= t^5 + 15t^4 + 20t^3 + 15t^2(2t + 1) + 6t(5t + 1) \\ &= t^5 + 15t^4 + 50t^3 + 45t^2 + 6t. \end{aligned} \tag{24}$$

Since $r(M) = 5$, $P_M(t)$ has degree at most 2 and gives a palindromic polynomial when added to (24). Thus, $P_{C_6}(t) = 5t^2 + 9t + 1$.

The Linear Coefficient

In Example 3.5, it was possible to find the linear coefficient of $P_{C_6}(t) = 5t^2 + 9t + 1$ directly from (22) and without computing $P_{M_a}(t)$. After we found $Z^*_M(t)$ in (24) explicitly, the linear coefficient of the desired Kazhdan-Lusztig polynomial was obtained as $Z^*_M(t)[t^4] - Z^*_M(t)[t] = 15 - 6 = 9$. In turn, these two coefficients come from the numbers of flats of the corresponding ranks and, as we shall see, are not affected by Kazhdan-Lusztig polynomials of other contraction matroids in (21). Indeed, Proposition 3.1 ensures that the constant term of $P_{M_a}(t)$ is 1, so $Z^*_M(t)[t] = 6$ regardless of the exact expression for $P_{M_a}(t)$. On the other hand, while $P_{M_{abcd}}(t)[t] = 1$, we also need to check that terms corresponding to other contraction matroids in (21) do not contribute to $Z_M^*(t)[t^4]$. Even though it is seen from (24) in Example 3.5, we wish to provide a more general argument using the degree restriction property of Kazhdan-Lusztig polynomials. In particular, for any $F \in L(M)$ with $r(F) \leqslant r(M) - 2$,

$$t^{r(F)}\deg(P_{M_F})(t) < r(F) + \frac{r(M_F)}{2} = \frac{r(M) + r(F)}{2} \leqslant r(M) - 1.$$

Hence, terms corresponding to flats of lower rank in the sum defining $Z_M^*(t)$ do not affect $Z_M^*(t)[t^{r(M)-1}]$. This general discussion leads to the formula for the linear coefficient of any Kazhdan-Lusztig polynomial.

Proposition 3.6 For any matroid M of rank $r = r(M)$, the linear coefficient of the corresponding Kazhdan-Lusztig polynomial $P_M(t)$ is

$$P_M(t)[t] = \left|L^{(r-1)}\right| - \left|L^{(1)}\right|, \tag{25}$$

where $L^{(k)} \subseteq L(M)$ is a collection of flats of rank k.

Again, this reasoning is possible only because computing linear coefficient of $P_M(t)$ involves coefficients of $Z_M^*(t)$ that are not affected by other terms in the sum defining $Z_M^*(t)$. Nevertheless, a similar approach can be applied to obtain the quadratic coefficient $P_M(t)[t^2]$ as the emerging interaction effects are still very limited. This, however, requires knowledge of the structure of contraction matroids in question.

Some Known Polynomials

As evident from Examples 3.3, 3.4 and 3.5, explicitly computing these Kazhdan-Lusztig polynomials relies heavily on symmetries of matroids in question. In fact, using much more advanced arguments, they were successfully computed for certain families of "nice" matroids.

In their original work, Elias et al. showed that for any two matroids $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$,

$$P_{M_1 \oplus M_2}(t) = P_{M_1}(t)P_{M_2}(t), \tag{26}$$

where $M_1 \oplus M_2 = (E_1 \cup E_2, \mathcal{I}_1 \cup \mathcal{I}_2)$ is a *direct sum* of M_1 and M_2 [1]. For graphic matroids $M(G_1)$ and $M(G_2)$, the direct sum matroid $M(G_1) \oplus M(G_2)$ is induced by the disjoint union $G_1 \sqcup G_2$. In particular, for disjoint graphs G_1 and G_2, $M(G_1 \cup G_2) = M(G_1) \oplus M(G_2)$. The same matroid is induced even if vertex sets $V(G_1)$ and $V(G_2)$ intersect by a single vertex, in which case we denote $G_1 \cup G_2$ by $G_1 \oplus G_2$ to emphasize the matroid relation $M(G_1) \oplus M(G_2) = M(G_1 \oplus G_2)$.

Gao et al. provided exact formulas for Kazhdan-Lusztig polynomials of uniform matroids $U_{m,d}$ and, in particular, solved the problem for any cycle C_{d+1} as it corresponds to a uniform matroid $U_{1,d}$ [2]. The formula for the i-th coefficient c_d^i of $P_{C_{d+1}}(t)$ is

$$c_d^i = \frac{1}{d-i}\binom{d+1}{i}\binom{d-i}{i+1}, \qquad 0 \leqslant i \leqslant \left\lfloor \frac{d-1}{2} \right\rfloor. \tag{27}$$

Note that our results $P_{C_4}(t) = 2t + 1$, $P_{C_5}(t) = 5t + 1$ and $P_{C_6}(t) = 5t^2 + 9t + 1$ are consistent with this formula.

In a recent study, Lu et al. had success with fan, wheel and whirl matroids [7]. A fan F_n is a graph given by a fan triangulation of a polygon on $n + 1$ vertices. A wheel W_n is again a graph on $n + 1$ vertices and consists of a cycle C_n, all vertices of which are connected to a single vertex outside of the cycle. Even though whirl matroid is defined via a wheel graph, it is not graphic in a formal sense. Examples of fan and wheel graphs are displayed in Figure 4 and formulas for the i-th coefficients f_n^i and w_n^i of $P_{F_n}(t)$ and $P_{W_n}(t)$, respectively, are given below.

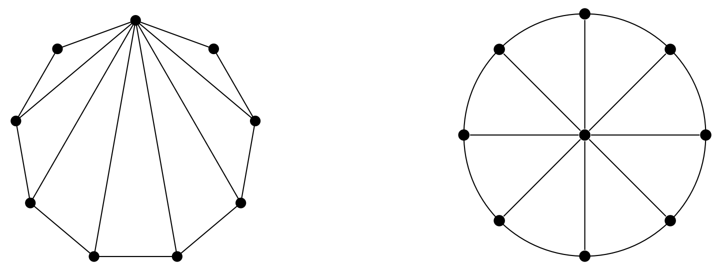

Figure 4: A fan graph F_8 (left) and a wheel graph W_8 (right).

$$f_n^i = \frac{1}{i+1}\binom{n-1}{i, i, n-2i-1},$$
$$w_n^i = \left(\frac{i+1}{n-i} + \frac{i}{n-i+1} - \frac{i}{n-i-1}\right)\binom{n}{i, i+1, n-2i-1}, \qquad 0 \leqslant i \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Gedeon computed Kazhdan-Lusztig polynomials of thagomizer matroids [3]. This class of graphic matroids arises from graphs that consist of n 3-cycles intersecting by a common edge. Formally, a thagomizer graph T_n is a complete bipartite graph $K_{2,n}$ with an edge between two distinguished vertices. Gedeon showed that the i-th coefficient t_n^i of $P_{T_n}(t)$ is the number of Dyck paths of n semilengths with i long ascents and has the following formula:

$$t_n^i = \frac{1}{n+1}\binom{n+1}{i}\sum_{j=2i}^{n}\binom{j-i-1}{i-1}\binom{n+1-i}{n-j}, \qquad 0 \leqslant i \leqslant \left\lfloor \frac{n}{2} \right\rfloor.$$

This last study inspired us to investigate Kazhdan-Lusztig polynomials of a *double-cycle* matroid, which will be the topic of the next section.

4 Double-Cycle Matroid

As opposed to a thagomizer graph, which is formed by a varying number of 3-cycles "glued" together by a particular edge, a double-cycle graph consists of two edge-intersecting cycles of arbitrary sizes.

Definition 4.1 A *double-cycle* $C_{m,n}$ $(m, n \geqslant 3)$ is a graph formed by two cycles, C_n and C_m, that intersect by a unique common edge e. Figure 5 exhibits $C_{6,5}$ as an example. A graphic matroid $M(C_{m,n})$ will be denoted by $M_{m,n}$.

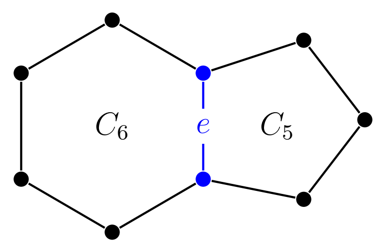

Figure 5: A double-cycle graph $C_{6,5}$.

In this section, we discover useful features of a double-cycle matroid that allow us to implement a computer program that calculates its Kazhdan-Lusztig polynomial. The generated data will reveal a connection between $P_{Mm,n}(t)$ and $P_{U_d}(t)$, which we state as a conjecture and prove in the next section.

Flats of a Double-Cycle Matroid $M_{m,n}$

We begin the exploration of a double-cycle matroid $M_{m,n}$ by classifying its flats. For convenience, partition the set of all flats of a given rank $1 \leqslant k \leqslant r(M_{m,n}) - 1$ into the following three subcollections:

$$L_1^{(k)} = \{F \in L(M_{m,n}) \mid e \notin F, k = r(F) = |F|\}, \tag{28}$$

$$L_2^{(k)} = \{F \in L(M_{m,n}) \mid e \in F, k = r(F) = |F|\}, \tag{29}$$

$$L_3^{(k)} = \{F \in L(M_{m,n}) \mid k = r(F) = |F| - 1\}. \tag{30}$$

Note that $L_1^{(k)}$ $L_2^{(k)}$ consist of flats that are forests with or without the edge e, while $L_3^{(k)}$ is a collection of flats that contain one of the cycles entirely. Hence, in particular, any flat in the third category $L_3^{(k)}$ must contain the edge e. Since any non-trivial flat $F \neq E(C_{m,n})$ satisfies $r(F) \leqslant |F| \leqslant r(F) + 1$, it is seen that every flat $F \in L(M_{m,n})$ of rank $1 \leqslant k \leqslant r(M_{m,n}) - 1$ belongs to one of these disjoint collections. A flat $F \in L_1^{(k)}$ is constructed by choosing some $j \leqslant m - 2$ edges from $C_m \backslash e$ and the remaining $k - j$

edges from $C_n \backslash e$. However, it is required that $k-j \leqslant n-2$ to ensure that the addition of e would not create a cycle. Thus,

$$\left|L_1^{(k)}\right| = \sum_{j=k-(n-2)}^{m-2} \binom{m-1}{j}\binom{n-1}{k-j}. \tag{31}$$

Remark 4.2 A closed form of the formula (31) can be obtained by considering all possible choices of k edges from $m-n-2$ (e must not be included) and then excluding the number of selections with all $m-1$ edges of $C_m \backslash e$ or all $n-1$ edges of $C_n \backslash e$. Hence, equivalently,

$$\left|L_1^{(k)}\right| = \binom{m+n-2}{k} - \binom{m-1}{k-n+1} - \binom{n-1}{k-m+1}. \tag{32}$$

The number of flats in $L_2^{(k)}$ is obtained using a similar reasoning adjusted only by the requirement of taking the edge e. Now, we are allowed to choose some $j \leqslant m-3$ edges from C_m and the remaining $k-j-1$ edges from C_n. Again, it is necessary to require that $k-j-1 \leqslant n-3$, so

$$\left|L_2^{(k)}\right| = \sum_{j=k-(n-2)}^{m-3} \binom{m-1}{j}\binom{n-1}{k-j-1}. \tag{33}$$

Remark 4.3 Again, an equivalent formula is obtained by first choosing any $k-1$ of $m+n-2$ edges (e is already included) and then subtracting the number of selections where at least $m-2$ edges come from $C_m \backslash e$ or at least $n-2$ edges come from $C_n \backslash e$.

$$\begin{aligned}\left|L_2^{(k)}\right| = \binom{m+n-2}{k-1} - \binom{m-1}{m-2}\binom{n-1}{k-m+1} - \binom{n-1}{k-m} \\ - \binom{m-1}{k-n+1}\binom{n-1}{n-2} - \binom{m-1}{k-n}.\end{aligned} \tag{34}$$

Finally, flats in $L_3^{(k)}$ are obtained by first choosing all edges of one of the cycles and then selecting the remaining edges from the other. Since the total number of selected edges must be $k+1$,

$$\left|L_3^{(k)}\right| = \binom{m-1}{k+1-n} + \binom{n-1}{k+1-m}. \tag{35}$$

When all flats are classified and counted, linear coefficient of the Kazhdan-Lusztig polynomial of any double-cycle matroid is found according to Proposition 3.6.

Theorem 4.4 Given a double-cycle matroid $M_{m,n}$ of rank $r = m+n-3$, the linear coefficient of its Kazhdan-Lusztig polynomial is

$$P_{M_{m,n}}(t)[t] = \sum_{i=1}^{3}\left|L_i^{(r-1)}\right| - \sum_{i=1}^{3}\left|L_i^{(1)}\right| = \binom{m+n-2}{2} - \binom{m+n-1}{1}. \tag{36}$$

Proof. This formula follows directly from Proposition 3.6 along with identities (32), (34) and (35) obtained above. □

Example: Kazhdan-Lusztig Polynomial of $M_{4,4}$

Consider $M = M_{4,4}$ of rank $r(M) = 5$. The degree restriction property of Kazhdan-Lusztig polynomials allows $P_{M_{4,4}}(t)$ to be quadratic, so the formula (36) has limited use. In order to obtain $Z^*_M(t)$, we only need to know the Kazhdan-Lusztig polynomials of contractions M_F when $r(F) = 1, 2$, because flats of larger rank give $P_{M_F}(t) = 1$ according to Proposition 3.2.

Flats of rank 1 are exactly the edges of $C_{4,3}$, however, the structure of a corresponding contraction matroid depends on the position of the contracted edge. There are exactly two possibilities: either an edge $f \neq e$ comes from either cycle C_4 (6 choices) or $f = e$. In each of these two cases, the resulting contraction matroids are $M_{4,3}$ and $M(C_3 \oplus C_3)$, respectively. Note that $r(M_{4,3}) = 4$, so the degree restriction property ensures that $P_{M_{4,3}}(t)$ is at most linear. Hence, $P_{M_{4,3}}(t) = 4t + 1$ according to (36). On the other hand,$P_{C_3 \oplus C_3}(t) = P_{C_3}(t)P_{C_3}(t) = 1$ as these polynomials must be trivial.

Any flat of rank 2 comes from either $L^{(2)}_1$ or $L^{(2)}_2$ as it cannot contain the entire cycle C_4. In the first case, a flat may be composed of two edges from the same cycle ($2 \cdot 3$ choices) or from different cycles ($3 \cdot 3$ choices). These two selections yield contractions $M(C_4)$ and $M_{3,3}$, respectively, while Examples 3.3, 3.4 as well as formula (36) provide their Kazhdan-Lusztig polynomials: $P_{C_4}(t) = 2t + 1$ and $P_{M_{3,3}}(t) = t + 1$. On the other hand, when a flat is of the second type, one edge (in addition to e) must be selected from the 6 remaining options. Regardless of the choice, the resulting contraction matroid is $M(C_3 \oplus P_1)$ and formula (26) gives $P_{C_3 \oplus P_1}(t) = P_{C_3}(t)P_{P_1}(t) = 1$.

Collecting all of the above Kazhdan-Lusztig polynomials, we finally write

$$\begin{aligned} Z^*_M(t) = t^5 + t^4 \sum_{i=1}^{3} \left| L^{(4)}_i \right| + t^3 \sum_{i=1}^{3} \left| L^{(3)}_i \right| &+ t^2 \left(9P_{M_{3,3}}(t) + 6P_{C_4}(t) + 6P_{C_3 \oplus P_1}(t)\right) \\ &+ t\left(6P_{M_{4,3}}(t) + P_{C_3 \oplus C_3}(t)\right) \\ &= t^5 + 15t^4 + 50t^3 + 45^2 + 7t. \end{aligned}$$

where we used (31), (33) and (35). It now follows that the desired Kazhdan-Lusztig polynomial is

$$P_{M_{4,4}}(t) = 5t^2 + 8t + 1. \tag{37}$$

It is tempting to design a computer program that will store a dictionary of Kazhdan-Lusztig polynomials of various double-cycle matroids and use it to compute larger examples with the Z^*-polynomial recursion. To achieve this, we must systematize the process in the above example. In particular, the algorithm will need to methodically identify all flats, count them and, most importantly, compute the corresponding contraction matroids. While flats are already thoroughly explored and their number is known (31, 33, 35), the last missing piece of information concerns contraction matroids, which we discuss now.

Contractions of $M_{m,n}$

In general, the three types of flats defined in (28), (29) and (30) induce three different structures of contraction matroids $(M_{m,n})_F$. Recall that $(M_{m,n})_F$ is induced by a graph $C_{m,n}$ with edges from F contracted. We now consider contractions of a double-cycle matroid with respect to flats from $L_1^{(k)}, L_2^{(k)}$ and $L_3^{(k)}$ separately.

Flats from $L_1^{(k)}$

Suppose a flat $F_1 \in L_1^{(k)}$ has j edges from C_m and $k-j$ edges from C_n. Then, contraction matroid $(M_{m,n})_{F_1}$ is usually given by a double-cycle graph with cycles of sizes $m-j$ and $n-k+j$ intersecting by the edge e. For example, consider $M_{8,6}$, a flat $F_1 \in L_1^{(4)}$ with $j = 3$ and the graph $C_{8,6}/F_1 = C_{5,5}$ of the corresponding contraction shown in Figure 6.

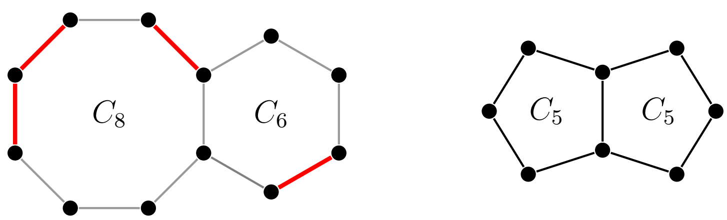

Figure 6: A flat F_1 (in red) and a corresponding contraction $C_{8,6}/F_1 = C_{5,5}$.

However, if j is too close to m ($j = m-2$) or $k-j$ is too close to n ($k-j = n-2$), contraction $C_{m,n}/F_1$ degenerates into a cycle or, if both inequalities hold, into a single edge. Figure 7 illustrates these possibilities by exhibiting $F_1 \in L_1^{(8)}$ of $M_{8,6}$ with $k-j = 4 = n-2$ on the left and $F_1' \in L_1^{(10)}$ of $M_{8,6}$ with both $j = 6 = m-2$ and $k-j = 4 = n-2$ on the right.

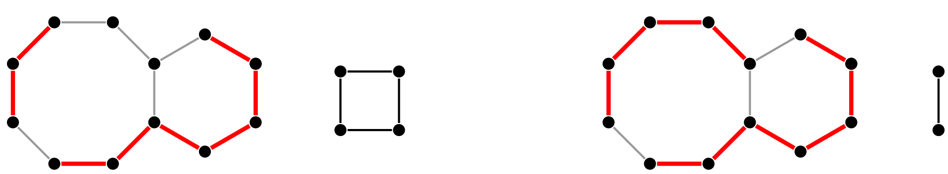

Figure 7: Flats F_1, F_1' (in red, left to right) with respective contractions $C_{8,6}/F_1 = C_4$ and $C_{8,6}/F_1' = P_1$

Flats from $L_2^{(k)}$

We take the same approach to understand contractions with respect to flats from $L_2^{(k)}$. The difference, however, is that flats of this type contain the edge e. Therefore, if $F_2 \in L_2^{(k)}$ has j edges from $C_m \backslash e$ and $k-j-1$ edges from $C_n \backslash e$, contraction matroid $(M_{m,n})_{F_2}$ is usually based on a direct sum of two cycles of sizes $m-j-1$ and $n+k-j-1$. Figure 8 shows a flat $F_2 \in L_2^{(4)}$ of $M_{8,6}$ with $j = 2$ and $k-j-1 = 1$ and the graph $C_5 \oplus C_4$ of the respective contraction matroid $(M_{8,6})_{F_2}$.

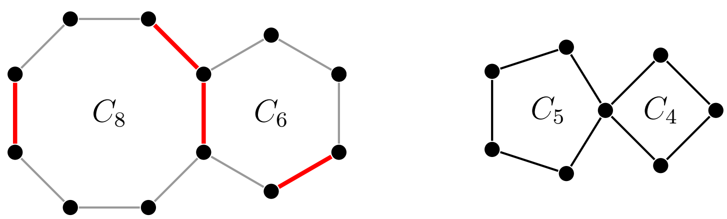

Figure 8: A flat F_2 (in red) and a corresponding contraction $C_{8,6}/F_2 = C_5 \oplus C_4$.

Again, for extreme values of j (either $j = m-3$ or $k-j = n-2$), contraction $C_{m,n}/F_2$ appears as a cycle with a path P_1 attached. In case both equations are satisfied, we obtain a path P_2 instead. These degenerate cases are illustrated in Figure 9 that shows $F_2 \in L_2^{(8)}$ of $M_{8,6}$ having $k-j = 4 = n-2$ and $F_2' \in L_2^{(9)}$ of $M_{8,6}$ with both $j = 5 = m-3-3$ and $k-j = 4-n-2$. These flats induce contractions based on graphs $C_3 \oplus P_1$ and P_2, respectively.

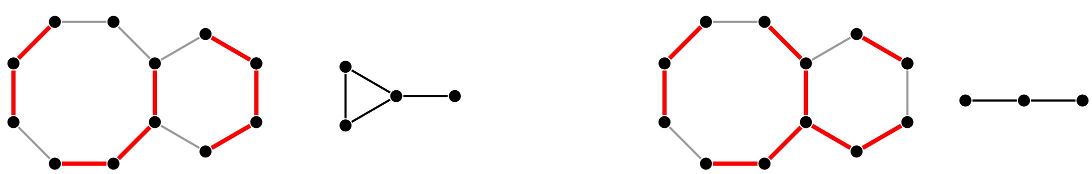

Figure 9: Flats F_2, F_2' (in red, left to right) with respective contractions $C_{8,6}/F_2 = C_3 \oplus P_1$ and $C_{8,6}/F_2' = P_2$

Flats from $L_3^{(k)}$

Finally, consider a flat $F_3 \in L_3^{(k)}$ containing the entire cycle C_m and some $k+1-m$ edges from $C_n \backslash e$. Contraction matroid with respect to F_3 is then based on a cycle of size $n-k+m-2$ unless $k+1-m = n-3$ (i.e. $k = r(M_{m,n})-1$), in which case we get just a single edge. Similar results hold when F_3 contains C_n instead of C_m. We illustrate both the general and the degenerate cases in Figure 10.
Thus, Kazhdan-Lusztig polynomial of any possible contraction of a double-cycle matroid $M_{m,n}$ can be found with a computer program based exclusively on already ob-

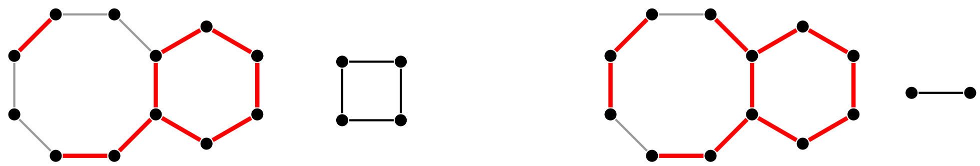

Figure 10: Flats $F_3 \in K_3^{(6)}$ and $F_3' \in L_3^{(10)}$ of $M_{8,6}$ (in red, left to right) with respective contractions $C_{8,6}/F_3 = C_4$ and $C_{8,6}/F_3' = P_1$

tained knowledge. Indeed, as seen from the above analysis, graphs that underlie contractions of $M_{m,n}$ are either double-cycles, cycles, paths or direct sums of cycles and/or paths. In the first case, the algorithm proceeds recursively until it reaches a contraction of a different type or a double-cycle matroid from the dictionary. For paths, Kazhdan-Lusztig polynomial is trivial ($P_{P_n}(t) = 1$), while formula (27) takes care of cycles. Finally, direct sums can be computed using relation $P_{G_1 \oplus G_2}(t) = P_{G_1}(t)P_{G_2}(t)$, which is a graphic case of (26).

The Algorithm and the Data

After all ingredients of the Z^*-polynomial (19) are understood, the algorithm is designed to compute Kazhdan-Lusztig polynomials of double-cycle matroids. For the implementation details, see Appendix A.

In light of the apparent connection between cycles and double-cycles, we compare the generated Kazhdan-Lusztig polynomials of these two types of matroids. As suggested by Propositions 3.1 and 3.6, the simplest patterns should appear in small coefficients, hence, we first focus on linear and quadratic coefficients (Tables 1, 2).

m/n	3	4	5	6	7	8	9	10	11	12	13	14
—	**0**	**2**	**5**	**9**	**14**	**20**	**27**	**35**	**44**	**54**	**65**	**77**
3	1	4	8	13	19	26	34	43	53	64	76	89
4	4	8	13	19	26	34	43	53	64	76	89	103
5	8	13	19	26	34	43	53	64	76	89	103	118
6	13	19	26	34	43	53	64	76	89	103	118	134
7	19	26	34	43	53	64	76	89	103	118	134	151
8	26	34	43	53	64	76	89	103	118	134	151	169
9	34	43	53	64	76	89	103	118	134	151	169	188
10	43	53	64	76	89	103	118	134	151	169	188	208

Table 1: Linear coefficients $P_{C_n}(t)[t]$ (in bold) and $P_{M_{m,n}}(t)[t]$.

It is expected that linear coefficients of Kazhdan-Lusztig polynomials of double-cycle matroids of equal rank are the same (see minor diagonals in Table 1) since formula

(36) depends on the sum $m+n$ rather than on the individual values. Note also that $P_{C_{3,n}}(t)[t]$ is listed as A034856 in the OEIS database [10]. A more useful observation, however, is that linear coefficients associated with double-cycle and cycle graphs differ by 1.

Theorem 4.5 Let C_d be a cycle on $d = m+n-2$ edges. Then,

$$P_{M_{m,n}}(t)[t] = P_{C_d}(t)[t] - 1. \tag{38}$$

Proof. Letting $d = m+n-3$ and $i = 1$ in formula (27), we obtain the linear coefficient of the Kazhdan-Lusztig polynomial of $U_{1,d}$ as

$$\begin{aligned} P_{U_{1,d}}(t)[t] &= \frac{1}{m+n-4}\binom{m+n-2}{1}\binom{m+n-4}{2} = \frac{(m+n-2)(m+n-5)}{2} \\ &= \frac{(m+n-3)(m+n-2)}{2} - \frac{2(m+n-2)}{2} \\ &= \binom{m+n-2}{2} - \binom{m+n-1}{1} + 1. \end{aligned} \tag{39}$$

The right side of (39) is exactly $P_{M_{m,n}}(t)[t]+1$ as stated in Theorem 4.4. □

The quadratic coefficients exhibit less straightforward patterns as seen from Table 2. Note that values that correspond to double-cycle matroids with $m, n \geqslant 4$ and fixed odd rank decrease by successive positive odd integers starting from the largest entry located on the main diagonal. For example, $P_{M_{8,8}}(t)[t^2] = 1337$, while $P_{M_{9,7}}(t)[t^2] = 1336$, $P_{M_{10,6}}(t)[t^2] = 1333$ etc. A similar trend is found for coefficients associated with double-cycle matroids of even rank as well.

m/n	3	4	5	6	7	8	9	10	11	12
—	**0**	**0**	**0**	**5**	**21**	**56**	**120**	**225**	**385**	**616**
3	0	0	3	16	47	106	205	358	581	892
4	0	5	19	51	111	211	365	589	901	1321
5	3	19	52	113	214	369	594	907	1328	1879
6	16	51	113	215	371	597	911	1333	1885	2591
7	47	111	214	371	598	913	1336	1889	2596	3483
8	106	211	369	597	913	1337	1891	2599	3487	4583
9	205	365	594	911	1336	1891	2600	3489	4586	5921
10	358	589	907	1333	1889	2599	3489	4587	5923	7529

Table 2: Quadratic coefficients $P_{C_n}(t)[t^2]$ (in bold) and $P_{M_{m,n}}(t)[t^2]$.

A far less obvious but much more important observation can be made while looking at Tables 1 and 2 simultaneously. We state it as a theorem below.

Theorem 4.6 Let $C_{1,d}$ be a cycle on $d = m + n - 2$ edges. Then,

$$P_{M_{m,n}}(t)[t^2] = P_{C_d}(t)[t^2] - P_{C_{m-1}}(t)[t] - P_{C_{n-1}}(t)[t] - 1. \quad (40)$$

This statement was verified by first writing out a general expression for the quadratic term $P_{M_{m,n}}(t)[t^2]$ and then simplifying it with Mathematica [6]. Based on Theorem 4.5 and the above result, we propose the following generalization.

Theorem 4.7 The Kazhdan-Lusztig polynomial of a double-cycle matroid $M_{m,n}$ is computed according to the formula

$$P_{C_{m,n}}(t) = P_{C_d}(t) - tP_{C_{n-1}}(t)P_{C_{m-1}}(t), \quad (41)$$

where $d = m + n - 2$.

Note that cycle C_d is obtained by deleting the common edge e from $C_{m,n}$, while C_{m-1} and C_{n-1} are the two parts of the contraction $C_{m,n}/e$ intersecting by a single vertex. Recall that cycles induce a special case of uniform matroids, Kazhdan-Lusztig polynomial of which is already known [2]. Thus, formula (41) allows us to write $P_{M_{m,n}}(t)$ for any double-cycle matroid $M_{m,n}$ explicitly.

Theorem 4.7 has been validated for higher coefficients using the designed algorithm. The formal proof, however, requires a more general discussion presented in the next section.

5 $\mathcal{Z}$-function of a Matroid

We now develop tools to prove some general statement about Kazhdan-Lusztig polynomials of arbitrary matroids, which will, in particular, imply Theorem 4.7. This result is stated below as a motivation for the coming section.

Theorem 5.1 Given matroid $M = (E, \mathcal{I})$ with a non-isthmus element $e \in E$, its Kazhdan-Lusztig polynomial $P_M(t)$ satisfies

$$P_M(t) = P_{M \backslash e}(t) - tP_{M/e}(t) + \sum_{F \in \mathcal{R}_M(e)} P_{M_F}(t)[t^{\tau(F)}]t^{\tau(F)+1}P_{M^{F \backslash e}}(t), \quad (42)$$

where $\tau(F) = \frac{cr(F)}{2} - \frac{1}{2}$ is an integer for flats in the subset $\mathcal{R}_M(e) \subseteq L(M)$, which will be formally introduced in Definition 5.13.

In some special cases, the sum on the right of equation (42) vanishes, providing a useful and often unsophisticated method of computing Kazhdan-Lusztig polynomials. Some applications of this technique to graphic matroids (and, in particular, to double-cycle matroids) will be discussed in the last section.

We begin with a few general facts that will help us throughout this section.

Lemma 5.2 Given a matroid $M = (E, \mathcal{I})$, subsets $A \subseteq B \subseteq E$ and an element $x \in E \backslash B$,

$$r(A \cup \{x\}) - r(A) \geqslant r(B \cup \{x\}) - r(B).$$

Proof. Apply the semimodularity property of r to subsets $A \cup \{x\}$ and B:

$$\begin{aligned} r((A \cup \{x\}) \cup B) + r((A \cup \{x\}) \cap B) &\leqslant r(A \cup \{x\}) + r(B), \\ r(B \cup \{x\}) + r(A) &\leqslant r(A \cup \{x\}) + r(B), \\ r(B \cup \{x\}) - r(B) &\leqslant r(A \cup \{x\}) - r(A). \end{aligned}$$

□

Lemma 5.3 If $M = (E, \mathcal{I})$ is a matroid with a non-isthmus element $e \in E$, then $r(M) = r(M \backslash e)$.

Proof. Assume $r(M \backslash e) < r(M)$ and consider any independent set A of M that does not contain e (one exists as e is non-isthmus). Apply Lemma 5.2 to $A, M \backslash e$, and e:

$$r(A \cup \{e\}) - r(A) \geqslant r\left((M \backslash e) \cup \{e\}\right) - r(M \backslash e) > 0,$$

hence, $r(A \cup \{e\}) > r(A) = |A|$. By the unit increase property of the rank function, $r(A \cup \{e\}) = |A| + 1 = |A \cup \{e\}|$, which implies that $A \cup \{e\}$ is independent. It follows that e is an isthmus, which contradicts the assumption. Therefore, we must conclude $r(M) = r(M \backslash e)$. □

The following definition introduces the $\mathcal{Z}$-function of a matroid—a central construction of this section.

Definition 5.4 Given a matroid M, let $\mathcal{Z}_M \colon L(M) \longrightarrow \mathbb{Z}[t]$ be a function that, for all $F \in L(M)$, is given by

$$\mathcal{Z}_M(F) = t^{r(F)} P_{M_F}(t). \tag{43}$$

This definition is quite similar to that of the Z-polynomial (18). In particular, the sum of the $\mathcal{Z}$-function over all flats $F \in L(M)$ is exactly $Z_M(t)$.

Theorem 5.5 The $\mathcal{Z}$-function has the following properties for all flats $F \in L(M)$:

(i) (Base) $\mathcal{Z}_M(M) = t^{r(M)}$;
(ii) (Divisibility) $t^{r(F)}$ divides $\mathcal{Z}_M(F)$, for all $F \in L(M)$;
(iii) (Palindromicity) $\sum\limits_{F \leqslant D} \mathcal{Z}_M(D)$ is palindromic of degree $r(M) + r(F)$. Equivalently,

$$\left(\sum_{F \leqslant D} \mathcal{Z}_M(D) \right)(t) = t^{r(M)+r(F)} \left(\sum_{F \leqslant D} \mathcal{Z}_M(D) \right)(t^{-1}); \tag{44}$$

(iv) (Degree Vanishing) If $r(M) > 0$ and $r(F) < r(M)$, $\deg\left(\mathcal{Z}_M(F)\right) < r(F) + \frac{cr(F)}{2}$.

Proof. Properties (i) and (ii) follow immediately from the definition of $\mathcal{Z}_M$. We now prove the remaining two properties:

(iii): According to the Definition 5.4,

$$t^{r(F)}\mathcal{Z}_{M_F}(D\backslash F) = t^{r(F)}t^{r_F(D\backslash F)}P_{(M_F)_{D\backslash F}}(t).$$

Using formula (14) for the contraction matroid's rank function and commutativity (8) of the contraction operation, we rewrite the above equation as

$$t^{r(F)}\mathcal{Z}_{M_F}(D\backslash F) = t^{r(D)}P_{M_D}(t).$$

With help of this result, we obtain

$$\sum_{F\leqslant D}\mathcal{Z}_M(D) = \sum_{F\leqslant D}t^{r(D)}P_{M_D}(t) = t^{r(F)}\sum_{F\leqslant D}\mathcal{Z}_{M_F}(D\backslash F) = t^{r(F)}Z_{M_F}(t).$$

Since Z-polynomial Z_{M_F} is palindromic of degree $r(M_F)$, the above equation can be rewritten as

$$\begin{aligned}\left(\sum_{F\leqslant D}\mathcal{Z}_M(D)\right)(t) &= t^{r(F)}Z_{M_F}(t) = t^{r(F)}t^{r_F(M_F)}Z_{M_F}(t^{-1})\\ &= t^{r(F)+r(M)}\left(t^{-r(F)}Z_{M_F}(t^{-1})\right)\\ &= t^{r(F)+r(M)}\left(\sum_{F\leqslant D}\mathcal{Z}_M(D)\right)(t^{-1}), \qquad (45)\end{aligned}$$

which is the desired equation (44).

(iv): Apply the degree restriction property from the definition of a Kazhdan-Lusztig polynomial and use formula (14) to derive

$$\deg\left(\mathcal{Z}_M(F)\right) = \deg\left(t^{r(F)}P_{M_F}(t)\right) < r(F) + \frac{r(M_F)}{2} = r(F) + \frac{cr(F)}{2},$$

proving the claim. ∎

Theorem 5.6 There is a unique function $L(M) \longrightarrow \mathbb{Z}[t]$ that satisfies all properties in Theorem 5.5.

Proof. Suppose there exists another function $\mathcal{Z}'_M\colon L(M) \longrightarrow \mathbb{Z}[t]$ that satisfies the four conditions above. We use induction on $cr(F)$ to show that $\mathcal{Z}_M(F) = \mathcal{Z}'_M(F)$ for all $F \in L(M)$. The base property (i) ensures that $\mathcal{Z}_M$ and $\mathcal{Z}'_M$ coincide on the only flat $M \in L(M)$ of zero corank. Now, assume that $\mathcal{Z}_M(F) = \mathcal{Z}'_M(F)$ for all $F \in L(M)$

with $cr(F) \leqslant k$. Let D be some flat of corank $k+1$. According to property (iii), the following polynomials are palindromic of degree $r(M) + r(D)$:

$$\sum_{D \leqslant F} \mathcal{Z}_M(F) = \mathcal{Z}_M(D) + \sum_{D < F} \mathcal{Z}_M(F), \tag{46}$$

$$\sum_{D \leqslant F} \mathcal{Z}'_M(F) = \mathcal{Z}'_M(D) + \sum_{D < F} \mathcal{Z}'_M(F). \tag{47}$$

Since $D < F$ implies $r(D) < r(F)$ and $cr(D) > cr(F)$, the inductive hypothesis yields

$$\sum_{D<F} \mathcal{Z}_M(F) = \sum_{D<F} \mathcal{Z}'_M(F).$$

In order to prove that $\mathcal{Z}_M(D) = \mathcal{Z}'_M(D)$, it suffices to note that $r(D) < r(F) \leqslant r(M)$ implies that their degrees are less than

$$r(D) + \frac{r(M) - r(D)}{2} = \frac{r(M) + r(D)}{2}$$

by the degree vanishing condition (iv). Therefore, we conclude $\mathcal{Z}_M(D) = \mathcal{Z}'_M(D)$ completing the induction. □

Definition 5.7 Given a deletion matroid $M\backslash e$, define a *pushforward* function $\delta\colon L(M) \longrightarrow L(M\backslash e)$ by $\delta(F) = F\backslash e$. Note that δ is well-defined because the lattice of flats of a deletion matroid is given by (9).

Identity (9) also ensures that for any $F \in L(M\backslash e)$, fiber $\delta^{-1}(F)$ consists of $F, F \cup \{e\}$, or both. We shall denote the least (inclusion-wise) element of this fiber by $\delta_0^{-1}(F)$.

Lemma 5.8 Given a matroid $M = (E, \mathcal{I})$, an element $e \in E$ and a flat $F \in L(M\backslash e)$,

$$r(\delta_0^{-1}(F)) = r(F). \tag{48}$$

Proof. In case $F \in \delta^{-1}(F)$, the desired statement follows trivially as $\delta_0^{-1}(F) = F$. Thus, assume that $F \notin L(M)$, so that $\delta_0^{-1}(F) = F \cup \{e\}$ and $r(\delta_0^{-1}(F)) = r(F \cup \{e\})$. Then, there must exist some $x \in E\backslash F$ such that $r(F) = r(F \cup \{x\})$. If $r(F) \neq r(F \cup \{e\})$, x must be different from e, which allows us to apply Lemma 5.2 to $F, F \cup \{e\}$ and x:

$$r((F \cup \{e\}) \cup \{x\}) - r(F \cup \{e\}) \leqslant r(F \cup \{x\}) - r(F) = 0,$$

which contradicts the assumption that $F \cup \{e\}$ is a flat of M. Hence, we must conclude $r(\delta_0^{-1}(F)) = r(F \cup \{e\}) = r(F)$. □

Theorem 5.9 Let $M\backslash e$ be a deletion matroid and let $D \in L(M\backslash e)$. Then, the preimage $\delta^{-1}([D, \hat{1}]) \subseteq L(M)$ is again an interval in $L(M)$. We will denote this interval by $\mathcal{J}_M(D)$.

Proof. We will show that $\delta^{-1}([D, \hat{1}]) = [\delta_0^{-1}(D), \hat{1}]$. Consider a flat $F \in \delta^{-1}([D, \hat{1}])$, so that $D \leqslant \delta(F) = F\backslash e$. In case $e \in F$, it is clear that both $D \subseteq F$ and $D \cup \{e\} \subseteq F$ hold, which implies $\delta_0^{-1}(D) \leqslant F \leqslant \hat{1}$. Otherwise, assume $e \notin F$ so that $D \leqslant \delta(F) = F$. If $\delta_0^{-1}(D) = D \cup \{e\}$ (i.e. $D \notin L(M)$), note that $r(D) = r(D \cup \{e\})$ by Lemma 5.8. Now, apply Lemma 5.2 to D, F and e:

$$r(F \cup \{e\}) - r(F) \leqslant r(D \cup \{e\}) - r(D) = 0,$$

which contradicts the fact that F is a flat of M. Hence, it must be that $\delta_0^{-1}(D) = D \leqslant F \leqslant \hat{1}$. This finishes the proof of $\delta^{-1}([E, \hat{1}]) \subseteq [(\delta_0^{-1}(E), \hat{1}]$. The reverse inclusion follows immediately by taking a flat $F \in [(\delta_0^{-1}(D), \hat{1}]$, which gives $D \subseteq \delta(F) = F\backslash e$. □

Corollary 5.10 Given a flat $D \in L(M\backslash e)$, the collection of fibers $\delta^{-1}(F)$ with $D \leqslant F$ partitions $\mathcal{J}_M(D)$.

Proof. Indeed, according to Theorem 5.9, the pushforward δ restricts to a well-defined surjective function on $\mathcal{J}_M(D)$ with codomain $[D, \hat{1}]$. □

Definition 5.11 Given a deletion matroid $M\backslash e$ equipped with a pushforward $\delta\colon L(M) \longrightarrow L(M\backslash e)$, define $\int \mathcal{Z}_M\colon L(M\backslash e) \longrightarrow \mathbb{Z}[t]$ as

$$\int \mathcal{Z}_M(D) := \sum_{F \in \delta^{-1}(D)} \mathcal{Z}_M(F). \tag{49}$$

Theorem 5.12 Given a deletion matroid $M\backslash e$ and a flat $D \in L(M\backslash e)$, polynomial

$$\sum_{D \leqslant F} \int \mathcal{Z}_M(F) \tag{50}$$

is palindromic of degree $r(D) + r(M)$.

Proof. According to Definition 5.4,

$$\sum_{D \leqslant F} \int \mathcal{Z}_M(F) = \sum_{D \leqslant F} \left(\sum_{C \in \delta^{-1}(F)} \mathcal{Z}_M(C) \right) = \sum_{C \in \mathcal{J}_M(D)} \mathcal{Z}_M(C), \tag{51}$$

where the last identity follows from Corollary 5.10. Since $\mathcal{J}_M(D) = [\delta_0^{-1}(D), \hat{1}]$ and $r(\delta_0^{-1}(D)) = r(D)$ (Lemma 5.8), the palindromicity property of $\mathcal{Z}_M$ implies that the polynomial in (51) is palindromic of degree

$$r\left(\delta_0^{-1}(D)\right) + r(M) = r(D) + r(M),$$

which is the desired statement. □

Definition 5.13 Given a matroid $M = (E, \mathcal{I})$ and an element $e \in E$, define a subset $\mathcal{R}_M(e) \subseteq L(M)$ of flats of M as

$$\mathcal{R}_M(e) := \{F \in L(M) \mid e \in F, F \backslash e \in L(M), cr(F) \text{ is odd}\}.$$

Additionally, we say that $F \in \mathcal{R}_M(e)$ *saturates* some $D \in L(M\backslash e)$ if $D = F\backslash e$.

Definition 5.14 For every flat $F \in \mathcal{R}_M(e)$, define S_F to be the following function on $L(M\backslash e)$:

$$S_F(D) = \begin{cases} \left(P_{M_F}(t)[t^{\tau(F)}]\right) t^{\tau(F)+1} \mathcal{Z}_{M^{F\backslash e}}(D) & \text{if } D \leqslant (F\backslash e) \\ 0 & \text{otherwise,} \end{cases} \tag{52}$$

where $\tau(F) = \frac{cr(F)}{2} - \frac{1}{2}$.

Lemma 5.15 Given a matroid $M = (E, \mathcal{I})$ with $r(M) > 0$, an element $e \in E$ and an unsaturated flat $D \in L(M\backslash e)$, $\int \mathcal{Z}_M(D)$ has the degree vanishing property at D, i.e.

$$\deg\left(\int \mathcal{Z}_M\left(D\right)\right) < r(D) + \frac{cr(D)}{2}. \tag{53}$$

Proof. This statement follows immediately by noting that for any unsaturated flat D, $\delta^{-1}(D) = \{D\}$ so that $\int \mathcal{Z}_M(D) = \mathcal{Z}_M(D)$. The degree vanishing condition of $\mathcal{Z}_M$ then completes the proof. □

We are finally ready to prove Theorem 5.1, which directly follows from a more general statement given in the next theorem.

Theorem 5.16 Given a matroid $M = (E, \mathcal{I})$ and a non-isthmus element $e \in E$, the following identity relates $\mathcal{Z}_{M\backslash e}$ and $\mathcal{Z}_M$:

$$\mathcal{Z}_{M\backslash e} = \int \mathcal{Z}_M - \sum_{F \in \mathcal{R}_M(e)} S_F, \tag{54}$$

Proof. Our strategy is to show that the polynomial on the right of equation (54) satisfies all properties of $\mathcal{Z}$-function on $L(M\backslash e)$ described in Theorem 5.5, in which case the uniqueness Theorem 5.6 would imply the desired equality. In light of the identity (10), rank function r of the original matroid M is used instead of $r^{(e)}$ when appropriate. Additionally, for convenience, we write $r(M)$ in place of $r(M\backslash e)$ since here e is non-isthmus (Lemma 5.3).

(i) **Base:** According to Definitions 5.4 and 5.11, the highest term of $\int \mathcal{Z}_M(M\backslash e)$ must be $t^{r(M)}$, which equals $t^{r(M\backslash e)}$ by Lemma 5.3. Therefore, it suffices to show that the degree of the sum in (54) evaluated at any flat $D < M\backslash e$ is less than $r(M\backslash e) = r(M)$. For each $D < M\backslash e$ and each $F \in L(M)$, if $D \notin [\hat{0}, F\backslash e]$, $S_F(D) = 0$ and the desired result follows trivially. Otherwise, the degree vanishing condition of $\mathcal{Z}_{M^{F\backslash e}}$ is applicable as $cr(F)$ is odd and hence nonzero:

$$\begin{aligned}\deg\left(S_F(D)\right) = \deg\left(t^{\tau(F)+1}\mathcal{Z}_{M^{F\backslash e}}(D)\right) &< \frac{cr(F)}{2} + \frac{1}{2} + \frac{r^{F\backslash e}(M^{F\backslash e}) + r^{F\backslash e}(D)}{2} \\ &< \frac{r(M) - r(F)}{2} + \frac{1}{2} + \frac{r(F\backslash e) + r(D)}{2} \\ &< \frac{r(M) + r(D)}{2},\end{aligned}$$

which establishes the base property as $r(D) < r(M)$. Note that we used $r(F) = r(F\backslash e) + 1$, which follows from Definition (5.13) and the unit-increase property of r applied to flats $F, F\backslash e \in L(M)$.

(ii) **Divisibility:** The divisibility property of $\mathcal{Z}_M$ together with Lemma 5.8 ensure that $\int \mathcal{Z}_M(D)$ is divisible by $t^{r(D)}$ for any flat $D \in L(M\backslash e)$. It now remains to show that $t^{r(D)}$ divides $S_F(D)$ for every $F \in \mathcal{R}_M(e)$ and each $D \in L(M\backslash e)$. In case that $D \notin [\hat{0}, (F\backslash e)]$, $S_F(D)$ is zero and hence divisible by $t^{r(D)}$. Otherwise, $D \leqslant (F\backslash e)$ so that $t^{r^{F\backslash e}(D)} = t^{r(D)}$ divides $\mathcal{Z}_{M^{F\backslash e}}$ and, therefore, $S_F(D)$ as well.

(iii) **Palindromicity:** We shall establish that for all flats $D \in L(M\backslash e)$, the following polynomial is palindromic of degree $r(D) + r(M\backslash e) = r(D) + r(M)$:

$$\begin{aligned}\sum_{D \leqslant C}\left(\int \mathcal{Z}_M(C) - \sum_{F \in \mathcal{R}_M(e)} S_F(C)\right) &= \sum_{D \leqslant C}\int \mathcal{Z}_M(C) - \sum_{D \leqslant C}\sum_{F \in \mathcal{R}_M(e)} S_F(C) \\ &= \sum_{D \leqslant C}\int \mathcal{Z}_M(C) - \sum_{F \in \mathcal{R}_M(e)}\sum_{D \leqslant C} S_F(C). \qquad (55)\end{aligned}$$

Recall from Theorem 5.12 that the first sum on the right side of the equation above is palindromic of the desired degree since $r(M) = r(M\backslash e)$. Now, for any flat $F \in \mathcal{R}_M(e)$, either $D \leqslant (F\backslash e)$ or $S_F(C) = 0$ for all $C \geqslant D$. Thus, we only consider flats D with $D \leqslant (F\backslash e)$:

$$\begin{aligned}\sum_{D \leqslant C} S_F(C) &= \sum_{D \leqslant C} P_{M_F}(t)[t^{\tau(F)}]t^{\tau(F)+1}\mathcal{Z}_{M^{F\backslash e}}(C) \\ &= P_{M_F}(t)[t^{\tau(F)}]t^{\tau(F)+1}\sum_{D \leqslant C}\mathcal{Z}_{M^{F\backslash e}}(C). \qquad (56)\end{aligned}$$

Third property of $\mathcal{Z}_{M^{F\backslash e}}$ (Theorem 5.5) ensures that the sum on the right side of equation (56) is palindromic of degree

$$r^{F\backslash e}(D) + r^{F\backslash e}(M^{F\backslash e}) = r(D) + r(F\backslash e).$$

Shifting this polynomial by $t^{\tau(F)+1}$ as in equation (56) increases its degree to

$$\begin{aligned} &r(D) + r(F\backslash e) + cr(F) + 1 \\ =\ &r(D) + r(F) - 1 + r(M) - r(F) + 1 \\ =\ &r(M) + r(D). \end{aligned}$$

Consequently, both sums in equation (55) are palindromic of matching degrees, so is their difference.

(iv) **Degree Vanishing:** We shall prove that for all flats $D < M\backslash e$,

$$\deg\left(\int \mathcal{Z}_M(D) - \sum_{F\in\mathcal{R}_M(e)} S_F(D)\right) < r(D) + \frac{cr(D)}{2}. \tag{57}$$

First, we prove that when F does not saturate D, $\deg(S_F(D)) < r(D) + \frac{cr(D)}{2}$. In case that $D \notin [\hat{0}, F\backslash e]$, $S_F(D) = 0$ and so the condition is trivially satisfied. Otherwise, if $D < F\backslash e$, the degree vanishing property of $Z_{M^{F\backslash e}}$ is applicable at D and gives

$$\begin{aligned} \deg\left(S_F(D)\right) &= \deg\left(P_{M_F}(t)[t^{\tau(F)}]t^{\tau(F)+1}\mathcal{Z}_{M^{F\backslash e}}(D)\right) \\ &< \frac{cr(F)}{2} + \frac{1}{2} + r^{F\backslash e}(D) + \frac{cr^{F\backslash e}(D)}{2} \\ &= \frac{r(M) - r(F) + 1}{2} + \frac{(r(F) - 1) + r(D)}{2} \\ &= r(D) + \frac{cr(D)}{2} \end{aligned} \tag{58}$$

Thus, $S_F(D)$ satisfies the degree vanishing condition provided that F does not saturate D. In particular, when $D \in L(M\backslash e)$ is unsaturated at all (e.g. D is of odd corank), $S_F(D)$ has the desired property for all $F \in \mathcal{R}_M(e)$. According to Lemma 5.15, the same holds for $\int \mathcal{Z}_M(E)$, which ensures inequality (57) for unsaturated flats.

Now, let $D \in L(M\backslash e)$ be saturated by some $F \in L(M)$ so that $F = D \cup \{e\}$. Since $F \in \mathcal{R}_M(e)$, $cr(F) = 2\tau(F) + 1$. Then, degree of any polynomial satisfying the degree vanishing condition at D must be less than

$$r(D) + \frac{cr(D)}{2} = r(D) + \frac{cr(F\backslash e)}{2} = r(D) + \tau(F) + 1. \tag{59}$$

The base and the degree vanishing properties of $\mathcal{Z}_{M^{F\backslash e}}$ show that the highest term of $S_F(D)$ is

$$t^{\tau(F)+1+r(F\backslash e)}P_{M_F}(t)[t^{\tau(F)}] = t^{\tau(F)+r(F)}P_{M_F}(t)[t^{\tau(F)}]. \tag{60}$$

The degree of the above monomial is $\tau(F) + r(F) = \tau(F) + r(D) + 1$, which is 1 more than allowed by the degree vanishing condition given in equation (59). Hence, this is the only term of $S_F(D)$ that violates the condition. We will now show that it is canceled out by $\int \mathcal{Z}_M(D)$. Note that $\delta^{-1}(D) = \{D \cup \{e\}, D\} = \{F, F\backslash e\}$ and so

$$\int \mathcal{Z}_M(D) = \mathcal{Z}_M(F) + \mathcal{Z}_M(F\backslash e) = t^{r(F)} P_{M_F}(t) + \mathcal{Z}_M(F\backslash e). \tag{61}$$

According to the degree vanishing property of Kazhdan-Lusztig polynomials and identity (14),

$$\deg(P_{M_F}(t)) < \frac{r_F(M_F)}{2} = \frac{cr(F)}{2} = \tau(F) + \frac{1}{2},$$

so the degree of $P_{M_F}(t)$ is at most $\tau(F)$. Then, equation (61) can be rewritten as

$$\begin{aligned}\int \mathcal{Z}_M(D) &= t^{r(F)} \left(P_{M_F}(t)[t^{\tau(F)}]t^{\tau(F)} + \sum_{i=0}^{\tau(F)-1} P_{M_F}(t)[t^i]t^i \right) + \mathcal{Z}_M(F\backslash e) \\ &= P_{M_F}(t)[t^{\tau(F)}]t^{r(F)+\tau(F)} + \left(\sum_{i=0}^{\tau(F)-1} P_{M_F}(t)[t^i]t^{r(F)+i} + \mathcal{Z}_M(F\backslash e) \right).\end{aligned}$$

The first summand in the above equation is the same as the highest term of $S_F(E)$ found in equation (60). Since $r(F) = r(D) + 1$ and $cr(F)$ is nonzero, it is easy to see that the other summand satisfies the degree vanishing condition at D. This allows us to conclude that

$$\int \mathcal{Z}_M(D) - \sum_{F \in \mathcal{R}_M(e)} S_F(D) \tag{62}$$

satisfies the degree vanishing condition for all saturated flats D as well.

It now follows that both functions in the equation (54) satisfy the four properties of a $\mathcal{Z}$-function described in Theorem 5.5. Hence, by the Uniqueness Theorem 5.6, these two functions must coincide. □

Proof of Theorem (5.1). In order to derive our main result of this section from the above Theorem, we need to evaluate both functions in equation (54) at the empty flat $\hat{0} \in L(M\backslash e)$, which yields

$$\mathcal{Z}_{M\backslash e}(\hat{0}) = \int \mathcal{Z}_M(\hat{0}) - \sum_{F \in \mathcal{R}_M(e)} P_{M_F}(t)[\tau(F)]t^{\tau(F)+1}\mathcal{Z}_{M^{F\backslash e}}(\hat{0}),$$

$$P_M(t) = P_{M\backslash e}(t) - tP_{M/e}(t) + \sum_{F \in \mathcal{R}_M(e)} P_{M_F}(t)[t^{\tau(F)}]t^{\tau(F)+1}P_{M^{F\backslash e}}(t). \qquad \square$$

In the next section, we derive an even more explicit method of computing Kazhdan-Lusztig polynomials using the above formula for some special graphic matroids.

6 Applications to Graphic Matroids

Formula (42) is particularly useful when the sum on the right evaluates to zero, leaving

$$P_M(t) = P_{M\backslash e}(t) - tP_{M/e}(t). \tag{63}$$

In this section, we discuss one way that it can happen in the context of graphic matroids.

Definition 6.1 A graph G is called *edge-separable* if it can be written as a union of two induced subgraphs $H, K \subseteq G$ that intersect by a single edge $e \in E(G)$ and both $H\backslash e$ and $K\backslash e$ are nonempty and connected. In this case, we write $G = H||_e K$.

Note the edge e in edge-separable graphs is non-isthmus, so formula (42) applies. The double-cycle graph $C_{m,n}$ is an example of an edge-separable graph. The two cycles C_m and C_n serve as subgraphs H and K from the above definition (Figure 11).

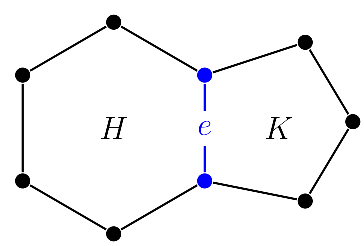

Figure 11: A double-cycle graph $C_{6,5}$ is edge-separable with $H = C_6$ and $K = C_5$.

Theorem 6.2 Given an edge-separable graph $G = H||_e K$, the corresponding Kazhdan-Lusztig polynomial $P_G(t)$ is computed according to formula

$$P_G(t) = P_{G\backslash e}(t) - tP_{H/e}(t)P_{K/e}(t), \tag{64}$$

Proof. Our strategy is to show that the sum in equation (42) of Theorem 5.1 is zero when G is edge-separable. Let $M = M(G)$ and consider any flat $F \in \mathcal{R}_M(e)$ so that $cr(F) = r_F(M_F) = 2\tau(F) + 1$. Additionally, we take advantage of the edge-separability of $G = H||_e K$ and partition F as $F = F_H \cup F_K \cup \{e\}$, where $F_H \subseteq H\backslash e$ and $F_K \subseteq K\backslash e$. Note also that F contains neither K nor H completely, for otherwise would contradict that $F\backslash e$ is a flat. Recall that the contraction matroid M_F is obtained from G by contracting all edges in F, i.e. $M_F = M(G/F)$. Since $e \in F$ and subgraphs $H' = H/F_H$, $K' = K/F_K$ are nonempty, $G' = G/F$ has a cut-vertex v_e (the leftover after contracting e) separating otherwise disjoint H' and K'. Hence, $M_F = M(H' \oplus K')$ and it follows that

$$2\tau(F) + 1 = r_F(M_F) = r_{H'}(H') + r_{K'}(K'), \tag{65}$$

where $r_{H'}$ and $r_{K'}$ denote rank functions of matroids $M(H')$ and $M(K')$, respectively [4]. Therefore, exactly one of $r_{H'}(H')$ and $r_{K'}(K')$ is odd. Without loss of generality, let these ranks be $2m$ and $2k+1$ for some integers m and k, respectively. Then, equation (65) becomes $2\tau(F)+1 = 2k+1+2m$, so that $\tau(F) = k+m$. Since $M(G') = M(H' \oplus K')$, we have $M(G') = M(H') \oplus M(K')$, whereby (26) implies that $P_{G'}(t) = P_{H'}(t)P_{K'}(t)$. Now, the degree vanishing property of Kazhdan-Lusztig polynomials is applicable to $P_{H'}(t)$ and $P_{K'}(t)$ as $M(H')$ and $M(K')$ have nonzero ranks:

$$\deg\left(P_{H'}(t)\right) < \frac{2m}{2} = m, \quad \deg\left(P_{K'}(t)\right) < \frac{2k+1}{2} = k + \frac{1}{2}. \tag{66}$$

It follows that the maximum possible degree of $P_{H'}(t)$ is $m-1$, and the maximum possible degree of $P_{K'}(t)$ is k. Thus, degree of $P_{G'}(t)$ is at most $m+k-1$, which is one less than $\tau(F)$. We conclude that for any $F \in \mathcal{R}_M(e)$, $P_{G/F}(t)[t^{\tau(F)}] = 0$, hence, the sum in equation (42) is zero when G is edge-separable. Finally, the same equation gives

$$P_G(t) = P_{G\backslash e}(t) - tP_{H/e}(t)P_{K/e}(t), \tag{67}$$

because contracting a single edge e in G yields a direct sum $H/e \oplus K/e$, whose Kazhdan-Lusztig polynomial is the product of polynomials of respective matroids. □

With the above result, it is trivial to check that our conjectured formula (41) for the Kazhdan-Lusztig polynomial of a double-cycle matroid is valid.

Proof of Theorem 4.7. As was noted earlier, any double-cycle $C_{m,n} = C_n||_e C_m$ is an edge-separable graph, so formula (64) applies. Since, $C_m/e = C_{m-1}$, $C_n/e = C_{n-1}$ and $C_{m,n}\backslash e = C_{m+n-2}$, it gives precisely the desired equation (41). □

We close this section by computing two examples to illustrate formula (64) at work.

Example 6.3 The Kazhdan-Lusztig polynomial of the double-cycle graph $G = C_{6,5}$ (Figure 11) is

$$\begin{aligned} P_G(t) &= P_{C_9}(t) - tP_{C_5}(t)P_{C_4}(t) = \\ &= \left(84t^3 + 120t^2 + 27t + 1\right) - t\left((2t+1)(5t+1)\right) \\ &= 74t^3 + 113t^2 + 26t + 1, \end{aligned} \tag{68}$$

where polynomials of uniform matroids $M(C_9)$, $M(C_5)$ and $M(C_4)$ are adopted from Table 1 of [1]. This result agrees with the output of our algorithm described in Appendix A.

Example 6.4 Consider a hexagon triangulation graph G as shown in Figure 12. This is an edge-separable graph with respect to one of its diagonals, which we denote by e. The two parts H and K are a 3-cycle and a 5-cycle with two diagonals incident to the vertex opposite to e, respectively.

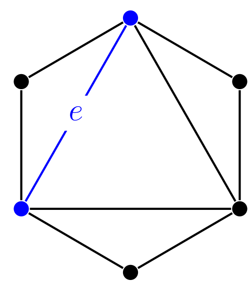

Figure 12: The edge-separable graph G given by a hexagon triangulation.

We apply formula (64) and get

$$\begin{aligned} P_G(t) &= P_{G\backslash e}(t) - tP_{H/e}(t)P_{K/e}(t) \\ &= P_{G\backslash e}(t) - t\left(P_{P_1}(t)\right)^2 P_{C_{3,3}}(t) = P_{G\backslash e}(t) - tP_{C_{3,3}}(t), \end{aligned} \tag{69}$$

since $H/e = P_1$, $K/e = C_{3,3} \oplus P_1$ and $P_{P_1}(t) = 1$. Graph $G\backslash e$ is again edge-separable with respect to any of the remaining diagonals. Let one of them be e' and $G' = G\backslash e = H'||_{e'}K'$, where $H' = C_3$ (below e') and K' is a 5-cycle with one diagonal (above e') as shown in Figure 13.

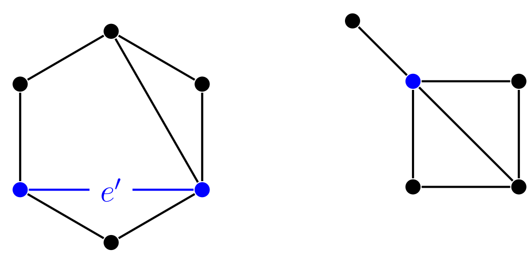

Figure 13: Graphs $G' = G\backslash e$ (left) and $K/e = C_{3,3} \oplus P_1$ (right).

Applying (64) to G',

$$\begin{aligned} P_{G'}(t) &= P_{G'\backslash e'}(t) - tP_{H'/e'}(t)P_{K'/e'}(t) \\ &= P_{G'\backslash e'}(t) - t\left(P_{P_1}(t)\right)^2 P_{C_{3,3}}(t) = P_{G'\backslash e'}(t) - tP_{C_{3,3}}(t), \end{aligned} \tag{70}$$

because $K'/e' = C_{3,3} \oplus P_1$ and $H'/e' = P_1$. Note that $G'\backslash e'$ is a double-cycle $C_{5,3}$ so that formula (64) gives

$$\begin{aligned} P_{G'\backslash e'}(t) &= P_{C_6}(t) - tP_{C_4}(t)P_{P_1}(t) \\ &= (5t^2 + 9t + 1) - t(2t + 1) = 3t^2 + 8t + 1, \end{aligned} \tag{71}$$

and, similarly, $PC_{3,3}(t) = t + 1$. Plugging these expressions back in equation (70) and then in (69), we finally get

$$P_{G'}(t) = (3t^2 + 8t + 1) - t(t + 1) = 2t^2 + 7t + 1, \tag{72}$$

$$P_G(t) = (2t^2 + 7t + 1) - t(t + 1) = t^2 + 6t + 1. \tag{73}$$

Therefore, Kazhdan-Lusztig polynomial of the hexagon triangulation in Figure 12 is t^2+6t+1. Note that this is an example of a non-double-cycle graph, Kazhdan-Lusztig polynomial of which was computed with formula (64).

7 Conclusion

In the first part of this study, we introduced a double-cycle graph and explored the structure of its matroid. When flats and contractions were classified and sufficiently understood, we designed an algorithm to generate examples of the corresponding Kazhdan-Lusztig polynomials. The data suggested a general formula (41) in terms of Kazhdan-Lusztig polynomials of uniform matroids. In the second part, we developed new constructions (e.g. the $\mathscr{Z}$-function, pushforward of a matroid) that were used to formulate and prove identity (42). This new result allows computing Kazhdan-Lusztig polynomial of a matroid via that of its deletion. Moreover, it degenerates to an even simpler expression when matroids in question possess a certain structure. For graphic matroids, we denoted this requirement by edge-separability, which, for example, is satisfied by double-cycle matroids.

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Appendix A The Algorithm

We present the Mathematica [6] program that uses the Z^*-polynomial technique to compute Kazhdan-Lusztig polynomial of a double-cycle matroid $M_{m,n}$ for given integers $m, n \geqslant 3$. The algorithm is designed to refer to a dynamic dictionary called **mainassoc** of the type **Association**, where it stores Kazhdan-Lusztig polynomials for different matroids already considered or encountered as intermediate steps. We begin with initializing the dictionary **mainassoc** and defining functions **NFlatsL1** and **NFlatsL2**, which compute individual terms of sums in (31) and (33) given a particular value of j, (recall that j measures the distribution of edges of a given flat between the two cycles of the graph). Additionally, define a **rk** variable that species the largest allowed rank of the input double-cycle matroid. Finally, we introduce a wrapper function **KLPolynomial** that takes cycle sizes m, n as inputs, which is intended for the user.

```
ClearAll
mainassoc = Association[]
rk = 15
Zerofunction[x_]:=0
NFlatsL1[n_, m_, j_, rank_]:=
Binomial[n − 1, j] ∗ Binomial[m − 1, rank − j]
NFlatsL2[n_, m_, j_, rank_]:=
Binomial[n − 1, j] ∗ Binomial[m − 1, rank − j − 1]
KLPolynomial[m_, n_]:=ComputeKLPolynomial[ZStarPolynomial[m, n]]
```

The two main functions, **ComputeKLPolynomial** and **ZStarPolynomial**, appear in the last definition. The first is responsible for extracting the Kazhdan-Lusztig polynomial from the Z^*-polynomial computed by the second.

```
ComputeKLPolynomial[zstar_]:=
Module[{degree = 0, Klpolynomial = Array[Zerofunction, rk, 0]},
For[j = 1, j ⩽ Length[zstar], j++, If[zstar[[j]] > 0, degree = j]];
Klpolynomial[[1]] = zstar[[degree]];
For[i = 2, i < degree/2 + 1, i++,
Klpolynomial[[i]] = zstar[[degree − i + 1]] − zstar[[i − 1]]];
Klpolynomial]
```

Computing the Z^*-polynomial requires several other methods, which we now define. First, the algorithm needs a function to insert new entries in a dictionary of polynomials and associate them with a key value. The next method provides this functionality.

```
CreateKLContractionsDatabaseEntry[newn_, newm_, type_]:=
Module[{zstar = Array[Zerofunction, rk, 1],
klpolynomial = Array[Zerofunction, rk, 1],
klpolynomial1 = Array[Zerofunction, rk, 1],
klpolynomial2 = Array[Zerofunction, rk, 1]},
If[type == 2, zstar = ZStarPolynomial[newn, newm];
klpolynomial = ComputeKLPolynomial[zstar];
AppendTo[mainassoc, {newn, newm, 2} → klpolynomial]];
If[type == 1,
klpolynomial1 = UniformPolynomial[newn];
klpolynomial2 = UniformPolynomial[newm];
klpolynomial[[1]] = 1;
For[i = 2, i < Length[klpolynomial], i++,
For[j = 1, j ⩽ i, j++, klpolynomial[[i]] =
klpolynomial[[i]] + klpolynomial1[[j]] * klpolynomial2[[i - j + 1]]]];
AppendTo[mainassoc, {newn, newm, 1} → klpolynomial]]; ]
```

Note that the above function uses method **UniformPolynomial**, which computes the Kazhdan-Lusztig polynomial associated with an n-cycle graph on any integer input $n \geqslant 3$. The definition of this function makes use of formula (27).

```
UniformPolynomial[n_]:=
Module[{Klpolynomial = Array[Zerofunction, rk, 0]},
For[i = 1, i ⩽ Floor[(n - 1)/2] + 1, i++,
Klpolynomial[[i]] = (1/(n - i)) * Binomial[n, i - 1] * Binomial[n - i, i]];
Klpolynomial]
```

Finally, we define the core function of this algorithm—**ZStarPolynomial**. It is responsible for computing the Z^*-polynomial of a matroid recursively by consulting the dictionary **mainassoc** and/or updating it upon encountering unseen matroids.

```
ZStarPolynomial[n_, m_]:=
Module[{Coefficients = Array[Zerofunction, rk],
rnk = n + m - 3, contributor = 0,
flatT1j = 0, flatT2j = 0}, Coefficients[[rnk]] = 1;
For[rnk = (n + m - 4), rnk > 0, rnk = rnk - 1,
For[contributor = rnk, contributor ⩾ Max[2 * rnk - n - m + 4, 1],
contributor = contributor - 1,
For[flatT1j = Max[contributor - (m - 2), 0],
flatT1j ⩽ n - 3, flatT1j++,
If[MemberQ[Keys[mainassoc],
{n - flatT1j - 1, m - contributor + flatT1j, 1}] == False,
CreateKLContractionsDatabaseEntry
[n - flatT1j - 1, m - contributor + flatT1j, 1]];
```

```
Coefficients[[rnk]] = Coefficients[[rnk]]+
(Lookup[mainassoc, {{n − flatT1j − 1, m − contributor + flatT1j, 1}}][[1]]
[[rnk − contributor + 1]]) ∗ NFlatsL2[n, m, flatT1j, contributor]];
For[flatT2j = contributor − (m − 2), flatT2j ⩽ n − 2, flatT2j++,
If[MemberQ[Keys[mainassoc], {n − flatT2j, m − contributor + flatT2j,
2}] == False,
CreateKLContractionsDatabaseEntry
[n − flatT2j, m − contributor + flatT2j, 2]];
Coefficients[[rnk]] = Coefficients[[rnk]]+
(Lookup[mainassoc, {{n − flatT2j, m − contributor + flatT2j, 2}}][[1]]
[[rnk − contributor + 1]]) ∗ NFlatsL1[n, m, flatT2j, contributor]];
Coefficients[[rnk]] = Coefficients[[rnk]]+
(UniformPolynomial[n − (contributor + 1 − m) − 1]
[[rnk − contributor + 1]])∗
Binomial[n − 1, contributor + 1 − m]+
(UniformPolynomial[m − (contributor + 1 − n) − 1]
[[rnk − contributor + 1]])∗
Binomial[m − 1, contributor + 1 − n]]];
Coefficients]
```

In the end, we show the performance of this algorithm by letting it compute Kazhdan-Lusztig polynomials of a few double-cycle matroids. Note that the output is given as an ordered list of coefficients, from low to high powers of t.

```
In[1]:= KLPolynomial[4, 4]
In[2]:= Length[mainassoc]
Out[1]={1, 8, 5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
Out[2]=15

In[3]:= KLPolynomial[6, 5]
In[4]:= Length[mainassoc]
Out[3]={1, 26, 113, 74, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
Out[4]=47

In[5]:= KLPolynomial[8, 8]
In[6]:= Length[mainassoc]
Out[5]={1, 76, 1337, 7406, 13426, 6566, 429, 0, 0, 0, 0, 0, 0, 0, 0}
Out[6]=139
```

Observe the increase in length of the dictionary **mainassoc** as we input larger matroids. To our delight, the first two Kazhdan-Lusztig polynomials agree with our own computations (37) and (68). This proves that, for matroids small enough, humans are no worse than computers.